Multi-Pomeron Exchange and the Universal Repulsion in Nuclear/Hyperonic Matter Triple-, Quadruple- and N-tuple-Pomeron Vertices

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I. INTRODUCTION

In these notes we derive the *effective two-body* baryon-baryon (nucleon-nucleon, hyperon-nucleon, hyperon-hyperon) force in matter from the triple-, quadruple-pomeron, and more general the N-tuple-pomeron, vertex.
General motivation: It was found by Nishizaki, Takatkska and Yamamoto [1] that the soft-core interactions tend to give a too low maximum for the neutron star mass, which is $M_{max} = 1.44 M_{\odot}$. To remedy this they add a repulsive universal TBF. This is all the more necessary since the discovery of the two-solar mass neutron stars [2, 3].

Like in Ref. [4], we consider the three- and also the four-body interactions between baryons as generated by the triple- and quadruple-pomeron vertex (see [5, 6] for references). Then, we integrate out one or two of the baryons to give an *effective two-body potential.*

In this note, we consider the triple-, quadruple-, and N-tuple-body interaction between baryons as a given by the triple-, quadruple-, and N-tuple-pomeron vertex. The framework we use is the description of the Pomeron with a scalar field $\sigma_P(x)$. It is a ghost-field in the sense that the propagator is *gaussian with (-)-sign.* So, the Pomeron does not propagate and gives only so-called 'contact' interactions with a Gaussian form factor. This is the picture used in [7] and also in the spirit of the Reggeon Field-theory formalism, see e.g. [6] and references.

Remarks: *(i) We give two derivations of the effective two-body potentials: with (a) Cartesian coordinates* $\mathbf{x}_i$*, and with Jacobian coordinates* $\mathbf{x}_\alpha$*. (ii) The multi-pomeron Lagrangians are without division by 3! and 4! for the triple- and quadruple-couplings respectively. As a consequence the effective two-body potentials get combinatorial factors 3! respectively 4!. (iii) The pomeron-vertices are defined with 'unrationalized couplings'* G_P, G_{3P}*, and* G_{4P} *for the pomeron-baryon, the triple-pomeron, and qudruple-pomeron couplings respectively. The 'rationalized couplings' are defined as* $g_P = G/\sqrt{4\pi}, g_{3P} = G_{3P}/(\sqrt{4\pi})^{3/2}$*, and* $g_{4P} = G_{4P}/(4\pi)^2$.

The content of these notes is as follows. In section II we review the two-body potential from pomeron-exchange. In section III the three-body potential is given and the effective two-body is derived, using in configuration space simple cartesian vectors for the position of the baryons. Similarly, in section IV and section V this is done for the four-body and N-body potentials. In section VI we discuus the triple- and quadruple couplings in connection with the Regge field-theory perspective. In Appendix A the derivation of the configuration space potentials is reviewed, within the context of the used normalization of the non-relativistic one-particle states. In Appendix B the three-body configuration-space potentials are derived using Jacobian coordinates for the baryons. Similarly in Appendix C for the four-body potentials. Finally, in Appendix D the Jacobian coordinates are described in more detail.

Literature: Nishizaki, Takatsuka, Yamamoto, P.T.P. 105 (2001); ibid 108 (2002).
A.B. Kaidalov & K.A. Ter-Materosyan, Nucl.Phys. B75 (1974).
Th.A. Rijken, Thesis, Nijmegen 1975 (unpublished).

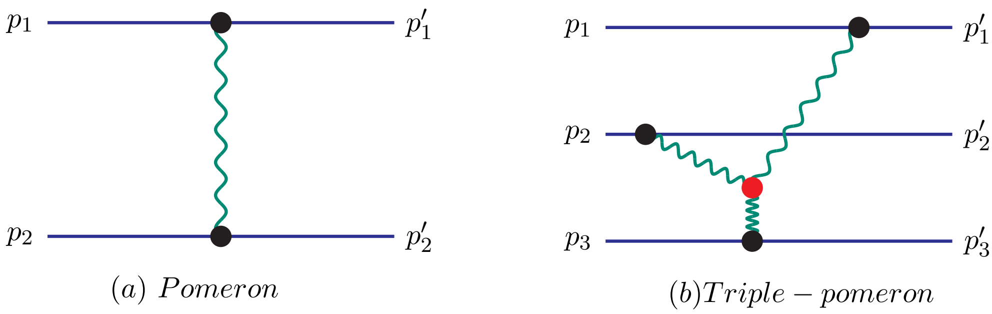

FIG. 1: *Pomeron and triple-pomeron exchange graphs*

Combinatorial factors: Associate $\sigma(x)$ with the Pomeron, and the BBP coupling

$$\mathcal{L}_{BBP} = g_P \left[\bar{\psi}(x)\psi(x)\right]\sigma(x). \tag{1.1}$$

The triple and quartic pomeron self-interactions we define as

$$\mathcal{L}_{PPP} = g_{3P}\ \sigma^3(x)\ ,\ \mathcal{L}_{PPPP} = g_{4P}\ \sigma^4(x). \tag{1.2}$$

a. Triple-pomeron exchange three-body force: 4th order diagram

$$\mathcal{M}_{3P} \sim \frac{1}{4!}\left[\mathcal{L}_{3P} + \mathcal{L}_{BBP}\right]^4 \Rightarrow 4 \times \frac{1}{4!}\ \mathcal{L}_{3P}\ \mathcal{L}^3_{BBP}$$
$$\rightarrow \text{Combinatorial factor diagram} : 4 \times \frac{1}{4!} \times 3! = 1.$$

b. Quartic-pomeron exchange four-body force: 5th order diagram

$$\mathcal{M}_{4P} \sim \frac{1}{5!}\left[\mathcal{L}_{3P} + \mathcal{L}_{BBP}\right]^5 \Rightarrow 5 \times \frac{1}{5!}\ \mathcal{L}_{4P}\ \mathcal{L}^4_{BBP}$$
$$\rightarrow \text{Combinatorial factor diagram} : 5 \times \frac{1}{5!} \times 4! = 1.$$

Conclusion: The Lagrangians in (1.2) give no extra combinatorial factors in the 3- and 4-body potential diagram.

II. TWO-BODY POTENTIAL FROM POMERON-EXCHANGE

Because of the universal coupling strength of the Pomeron to Baryons, we can restrict ourselves to nucleons, without loss of generality. We start from the pomeron-interaction.

The Lagrangian and the propagator, we take as [8]

$$\mathcal{L}_P = G_P \bar{\psi}(x)\psi(x)\ \sigma_P(x) \tag{2.1a}$$

$$\Delta_F^P(k^2) = \exp(-\mathbf{k}^2/4m_P^2)/\mathcal{M}^2\ , \tag{2.1b}$$

where the scaling mass $\mathcal{M} = 1$GeV. Then, the matrix element for the graph of Fig. 1 in the CM-system is given by [9], see Appendix A,

$$\begin{aligned} M_P(p_1', p_2'; p_1, p_2) &= G_P^2\ [\bar{u}(p')u(p)]\,[\bar{u}(-p')u(-p)] \cdot \Delta_F^P[(p'-p)^2] \\ &\approx G_P^2 \exp\left(-\mathbf{k}^2/4m_P^2\right)/\mathcal{M}^2, \end{aligned} \tag{2.2}$$

where we used the CM-momenta, i.e. $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$, and $\mathbf{p}_1' = -\mathbf{p}_2' = \mathbf{p}'$. We also introduced $\mathbf{k} = \mathbf{p}' - \mathbf{p}$. Then, the potential in configuration space is given by

$$\begin{aligned} V_P(r_{12}) &= \int \frac{d^3k}{(2\pi)^3}\ e^{i\mathbf{k}\cdot\mathbf{x}} M_P(\mathbf{p}', \mathbf{p})\delta(\mathbf{k} - \mathbf{p}' + \mathbf{p}) \\ &= \frac{G_P^2}{4\pi}\frac{4}{\sqrt{\pi}}\frac{m_P^3}{\mathcal{M}^2}\ \exp\left(-m_P^2 r_{12}^2\right)\ . \end{aligned} \tag{2.3}$$

For the volume integral we get

$$I_V^{(2)} = \int d^3r_{12} V_P(r_{12}) = G_P^2/\mathcal{M}^2\ . \tag{2.4}$$

III. THREE-BODY POTENTIAL FROM THE TRIPLE-POMERON VERTEX

For the triple-pomeron vertex we take the Lagrangian

$$\mathcal{L}_{PPP} = G_{3P}\mathcal{M}\ \sigma_P^3(x) \tag{3.1}$$

Then, the matrix element for the graph of 1 is given by

$$\begin{aligned} M_{3P}(p_1', p_2', p_3'; p_1, p_2, p_3) &= G_{3P}G_P^3\ \mathcal{M}\ \Pi_{i=1}^3 \left\{[\bar{u}(p_i')u(p_i)]\,\Delta_F^P[(p_i'-p_i)^2]\right\} \\ &\approx G_{3P}G_P^3\ \mathcal{M}\ \Pi_{i=1}^3 \Delta_F^P[(p_i'-p_i)^2]\ . \end{aligned} \tag{3.2}$$

The corresponding three-body potential in configuration space is given by

$$\begin{aligned} V(\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_3'; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) &= \Pi_{i=1}^3 \left[\int \frac{d^3p_i'}{(2\pi)^3}\frac{d^3p_i}{(2\pi)^3} \cdot e^{-i\left(\mathbf{p}_i'\cdot\mathbf{x}_i' - \mathbf{p}_i\cdot\mathbf{x}_i\right)}\right] \\ &\times M_{3P}(p_1', p_2', p_3'; p_1, p_2, p_3)\ \delta\left(\sum \mathbf{p}_i' - \sum \mathbf{p}_i\right). \end{aligned} \tag{3.3}$$

Introducing now the combinations

$$\mathbf{q}_i = \frac{1}{2}\left(\mathbf{p}_i' + \mathbf{p}_i\right)\quad,\quad \mathbf{k}_i = \mathbf{p}_i' - \mathbf{p}_i, \tag{3.4a}$$

$$\mathbf{p}_i' = \mathbf{q}_i + \frac{1}{2}\mathbf{k}_i\quad,\quad \mathbf{p}_i = \mathbf{q}_i - \frac{1}{2}\mathbf{k}_i\ . \tag{3.4b}$$

Then, we have that $d^3p_i'd^3p_i = d^3q_id^3k_i$, and

$$\mathbf{p}_i' \cdot \mathbf{x}_i' - \mathbf{p}_i \cdot \mathbf{x}_i = \mathbf{q}_i \cdot (\mathbf{x}_i' - \mathbf{x}) + \frac{1}{2}\mathbf{k}_i \cdot (\mathbf{x_i}' + \mathbf{x}_i) \tag{3.5}$$

The q_i-integrations can be done immediately,

$$\int d^3q_i \ \exp\{\mathbf{q}_i \cdot (\mathbf{x}_i' - \mathbf{x}_i)\} = (2\pi)^3\delta(\mathbf{x}_i' - \mathbf{x}_i).$$

After this we get for the three-body potential

$$V(\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_3'; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \equiv V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)\delta(\mathbf{x}_1' - \mathbf{x}_1)\delta(\mathbf{x}_2' - \mathbf{x}_2)\delta(\mathbf{x}_3' - \mathbf{x}_3) \tag{3.6a}$$

$$V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = G_{3P}G_P^3 \ \left[\Pi_{i=1}^3 \int \frac{d^3k_i}{(2\pi)^3} \ e^{-i\mathbf{k}_i\cdot\mathbf{x}_i}\right] \cdot \delta\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3\right) \cdot$$
$$\times \exp\left(-\mathbf{k}_1^2/4m_P^2\right) \exp\left(-\mathbf{k}_2^2/4m_P^2\right) \exp\left(-\mathbf{k}_3^2/4m_P^2\right) \cdot \mathcal{M}^{-5} \tag{3.6b}$$

where the Pomeron propagator $\Delta_F^P[k^2]$ given in Eq. (2.1b) is used.

A. The triple pomeron effective two-body potential

To obtain the effective two-body potential in a baryonic medium, we integrate over the coordinate $\mathbf{x}_3$ of the third nucleon. From (3.6b) it is evident that this gives a factor $(2\pi)^3\delta(\mathbf{k}_3)$. Using this we get from (3.6b) the two-body potential

$$V_{eff}(\mathbf{x}_1, \mathbf{x}_2) = \rho_{NM} \int d^3x_3 \ V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \tag{3.7a}$$

$$\begin{aligned} V_{eff}(\mathbf{x}_1, \mathbf{x}_2) &= G_{3P}G_P^3 \frac{\rho_{NM}}{\mathcal{M}^5} \cdot \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{-i\mathbf{k}_1\cdot\mathbf{x}_1} e^{-i\mathbf{k}_2\cdot\mathbf{x}_2} \cdot \\ &\quad \times\delta(\mathbf{k}_1 + \mathbf{k}_2) \exp\left(-\mathbf{k}_1^2/4m_P^2\right) \exp\left(-\mathbf{k}_2^2/4m_P^2\right) \\ &= G_{3P}G_P^3 \frac{\rho_{NM}}{\mathcal{M}^5} \cdot \int \frac{d^3k_1}{(2\pi)^6} e^{-i\mathbf{k}_1\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \cdot \exp\left(-\mathbf{k}_1^2/2m_P^2\right) \\ &= g_{3P}g_P^3 \frac{\rho_{NM}}{\mathcal{M}^5} \cdot \frac{8}{4\pi} \frac{4}{\sqrt{\pi}} \left(\frac{m_P}{\sqrt{2}}\right)^3 \ \exp\left(-\frac{1}{2}m_P^2 r_{12}^2\right). \end{aligned} \tag{3.7b}$$

In the last expression we introduced the *rationalized couplings*

$$g_P = G_P/\sqrt{4\pi} \ , \ g_{3P} = G_{3P}/(4\pi)^{3/2}. \tag{3.8}$$

Note that

(i) g_P is the Pomeron parameter in the Nijmegen potential program and papers.

(ii) result (3.7b) should be multiplied by the combinatorial factor: 3!

From (3.7b) one sees that if $g_{3P} > 0$ this gives repulsion in a few/many-body system.

Comparing formula (3.7b) with formula **(8.3)** in the ESC08c paper [10]

$$V_{eff}(\mathbf{x}_1, \mathbf{x}_2) \;=\; g'_{3P} g_P^3 \frac{\rho_{NM}}{\mathcal{M}^5} \cdot \frac{1}{4\pi} \frac{4}{\sqrt{\pi}} \left(\frac{m_P}{\sqrt{2}}\right)^3 \,\exp\left(-\frac{1}{2} m_P^2 r_{12}^2\right),$$

shows that $g'_{3P} = 8 g_{3P}$.

Now, one has that

$$\rho_{NM} = \frac{2 p_F^3}{3\pi^2}\;,\;\rho_0 = \frac{p_F^3}{6\pi^2}\;,\;\rho_{NM} = 4\rho_0\;. \tag{3.9}$$

The volume integral of V_{eff} is

$$I_{V,eff} = g'_{3P} g_P^3 \frac{\rho_{NM}}{\mathcal{M}^5} \cdot \frac{4}{\sqrt{\pi}} = g'_{3P} g_P^3 \frac{2}{3\pi^2} \left(\frac{p_F}{\mathcal{M}}\right)^3 \cdot \frac{1}{\mathcal{M}^2} \cdot \frac{4}{\sqrt{\pi}} \tag{3.10}$$

IV. FOUR-BODY POTENTIAL FROM THE QUADRUPLE-POMERON VERTEX

For the quadruple-pomeron vertex we take the Lagrangian

$$\mathcal{L}_{4P} \;=\; G_{4P}\; \sigma_P^4(x) \tag{4.1}$$

Then, the matrix element for the graph of 1 is given by

$$\begin{aligned} M_{4P}(p'_1, p'_2, p'_3, p'_4; p_1, p_2, p_3, p_4) \;&=\; G_{4P} G_P^4\; \Pi_{i=1}^4 \left\{ [\bar{u}(p'_i) u(p_i)]\, \Delta_F^P[(p'_i - p_i)^2] \right\} \\ &\approx\; G_{4P} G_P^4\; \Pi_{i=1}^4 \Delta_F^P[(p'_i - p_i)^2]\;. \end{aligned} \tag{4.2}$$

The corresponding four-body potential in configuration space is given by

$$\begin{aligned} V(\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}'_4; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \Pi_{i=1}^4 &\left[\int \frac{d^3 p'_i}{(2\pi)^3} \int \frac{d^3 p_i}{(2\pi)^3} e^{-i\left(\mathbf{p}'_i \cdot \mathbf{x}'_i - \mathbf{p}_i \cdot \mathbf{x}_i\right)} \right] \cdot \\ &\times M_{4P}(p'_1, p'_2, p'_3, p'_4; p_1, p_2, p_3, p_4)\; \delta\left(\sum_{i=1}^4 \mathbf{p}'_i - \sum_{i=1}^4 \mathbf{p}_i \right). \end{aligned} \tag{4.3}$$

Introducing now the combinations

$$\mathbf{q}_i = \frac{1}{2}\left(\mathbf{p}'_i + \mathbf{p}_i\right) \quad,\quad \mathbf{k}_i = \mathbf{p}'_i - \mathbf{p}_i\;,\;\text{or} \tag{4.4a}$$

$$\mathbf{p}'_i = \mathbf{q}_i + \frac{1}{2}\mathbf{k}_i \quad,\quad \mathbf{p}_i = \mathbf{q}_i - \frac{1}{2}\mathbf{k}_i\;. \tag{4.4b}$$

Then, we have that $d^3 p'_i d^3 p_i = d^3 q_i d^3 k_i$, and

$$\mathbf{p}'_i \cdot \mathbf{x}'_i - \mathbf{p}_i \cdot \mathbf{x}_i = \mathbf{q}_i \cdot (\mathbf{x}'_i - \mathbf{x}_i) + \frac{1}{2}\mathbf{k}_i \cdot (\mathbf{x_i}' + \mathbf{x}_i) \tag{4.5}$$

Again, the q_i-integrations can be done immediately, leading to the four-body potential

$$V(\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}'_4; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \equiv V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)\Pi_{i=1}^4 \delta(\mathbf{x}'_i - \mathbf{x}_i), \tag{4.6a}$$

$$V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = G_{4P} G_P^4 \ \Pi_{i=1}^4 \left\{ \int \frac{d^3k_i}{(2\pi)^3} \ e^{-i\mathbf{k}_i \cdot \mathbf{x}_i} \right\} \cdot \delta\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4\right) \cdot$$
$$\times \exp\left(-\mathbf{k}_1^2/4m_P^2\right) \exp\left(-\mathbf{k}_2^2/4m_P^2\right) \exp\left(-\mathbf{k}_3^2/4m_P^2\right) \exp\left(-\mathbf{k}_4^2/4m_P^2\right) \cdot \mathcal{M}^{-8} \ , \tag{4.6b}$$

A. The quadruple effective two-body potential

To obtain the effective two-body potential in a baryonic medium, we integrate over the coordinate $\mathbf{x}_3$ and $\mathbf{x}_4$ of the third and fourth nucleon. From (4.6b) it is evident that this gives the factors $(2\pi)^3\delta(\mathbf{k}_3)$ and $(2\pi)^3\delta(\mathbf{k}_4)$. Using this we get from (4.6b the two-body potential

$$\begin{aligned} V_{eff}(\mathbf{x}_1, \mathbf{x}_2) &= \rho_{NM}^2 \int d^3x_3 \int d^3x_4 \ V(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) \\ V_{eff}(\mathbf{x}_1, \mathbf{x}_2) &= G_{4P} G_P^4 \frac{\rho_{NM}^2}{\mathcal{M}^8} \cdot \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{-i\mathbf{k}_1\cdot\mathbf{x}_1} e^{-i\mathbf{k}_2\cdot\mathbf{x}_2} \cdot \\ &\quad \times \delta(\mathbf{k}_1 + \mathbf{k}_2) \exp\left(-\mathbf{k}_1^2/4m_P^2\right) \exp\left(-\mathbf{k}_2^2/4m_P^2\right) \\ &= G_{4P} G_P^4 \frac{\rho_{NM}^2}{\mathcal{M}^8} \cdot \int \frac{d^3k_1}{(2\pi)^6} e^{-i\mathbf{k}_1\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \cdot \exp\left(-\mathbf{k}_1^2/2m_P^2\right) \\ &= 8 g_{4P} g_P^4 \frac{\rho_{NM}^2}{\mathcal{M}^8} \cdot \frac{4}{\sqrt{\pi}} \left(\frac{m_P}{\sqrt{2}}\right)^3 \exp\left(-\frac{1}{2} m_P^2 r_{12}^2\right) . \end{aligned} \tag{4.7a}$$

Again, we introduced in the last line the *rationalized 4-point coupling* $g_{4P} = G_{4P}/(4\pi)^2$, similar to the rationalized 3-point coupling g_{3P}.
Note that the result (4.7a) should be multiplied by the combinatorial factor 4!
From (4.7a) it follows that if $g_{4P} > 0$ this gives repulsion in a few/many-body system.
Now, one has that

$$\rho_{NM} = \frac{2p_F^3}{3\pi^2} \ , \ \rho_0 = \frac{p_F^3}{6\pi^2} \ , \ \rho_{NM} = 4\rho_0 \ . \tag{4.8}$$

The volume integral of V_{eff} is

$$I_{V,eff} = g'_{4P} g_P^4 \frac{\rho_{NM}^2}{\mathcal{M}^8} = g'_{4P} g_P^4 \frac{4}{9\pi^4} \left(\frac{p_F}{\mathcal{M}}\right)^6 \cdot \frac{1}{\mathcal{M}^2} \ . \tag{4.9}$$

V. N-BODY POTENTIAL FROM THE N-TUPLE-POMERON VERTEX

The work of the foregoing sections is easily generalized to the case of an N-tuple-pomeron vertex. For the N-tuple-pomeron vertex we take the Lagrangian

$$\mathcal{L}_N \ = \ G_P^{(N)} \mathcal{M}^{4-N} \ \sigma_P^N(x) \tag{5.1}$$

The N-body potential is

$$V(\mathbf{x}_1',\ldots,\mathbf{x}_N';\mathbf{x}_1,\ldots,\mathbf{x}_N) \equiv V(\mathbf{x}_1,\ldots,\mathbf{x}_N)\Pi_{i=1}^N \delta(\mathbf{x}_i'-\mathbf{x}_i), \tag{5.2a}$$

$$V(\mathbf{x}_1,\ldots,\mathbf{x}_N) = G_P^{(N)} G_P^N\ \Pi_{i=1}^N \left\{\int \frac{d^3k_i}{(2\pi)^3}\ e^{-i\mathbf{k}_i\cdot\mathbf{x}_i}\right\}\cdot\delta\left(\mathbf{k}_1+\mathbf{k}_2+\ldots+\mathbf{k}_N\right)\cdot$$
$$\times\exp\left(-\mathbf{k}_1^2/4m_P^2\right)\exp\left(-\mathbf{k}_2^2/4m_P^2\right)\ldots\exp\left(-\mathbf{k}_N^2/4m_P^2\right)\cdot\mathcal{M}^{4-3N}\ , \tag{5.2b}$$

Similarly to the section III the effective two-body potential in a baryonic medium is obtained by integrating over the coordinates $\mathbf{x}_3$, ..., $\mathbf{x}_N$ of the nucleons (baryons). From (5.2b) it is evident that this gives the factors $(2\pi)^3\delta(\mathbf{k}_3)$ .... $(2\pi)^3\delta(\mathbf{k}_4)$. Using this we get from (5.2b) the two-body potential

$$\begin{aligned}
V_{eff}^{(N)}(\mathbf{x}_1,\mathbf{x}_2) &= \rho_{NM}^{N-2}\int d^3x_3\ \ldots \int d^3x_N\ V(\mathbf{x}_1,\mathbf{x}_2,\ldots\mathbf{x}_N) \\
V_{eff}^{(N)}(\mathbf{x}_1,\mathbf{x}_2) &= G_P^{(N)}G_P^N\frac{\rho_{NM}^{N-2}}{\mathcal{M}^{3N-4}}\cdot\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}e^{-i\mathbf{k}_1\cdot\mathbf{x}_1}e^{-i\mathbf{k}_2\cdot\mathbf{x}_2}\cdot \\
&\quad\times\delta(\mathbf{k}_1+\mathbf{k}_2)\exp\left(-\mathbf{k}_1^2/4m_P^2\right)\exp\left(-\mathbf{k}_2^2/4m_P^2\right) \\
&= G_P^{(N)}G_P^N\frac{\rho_{NM}^{N-2}}{\mathcal{M}^{3N-4}}\cdot\int\frac{d^3k_1}{(2\pi)^6}e^{-i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)}\cdot\exp\left(-\mathbf{k}_1^2/2m_P^2\right) \\
&= (4\pi)^{(N-4)/2}g_P^{(N)}g_P^N\frac{\rho_{NM}^{N-2}}{\mathcal{M}^{3N-4}}\cdot\frac{8}{\pi\sqrt{\pi}}\cdot\left(\frac{m_P}{\sqrt{2}}\right)^3\ \exp\left(-\frac{1}{2}m_P^2r_{12}^2\right)\ .
\end{aligned} \tag{5.3}$$

Therefore, if $g_P^{(N)} > 0$ this gives repulsion in a few/many-body system. In (5.3) we introduced the *rationalized coupling* $g_P^{(N)} = G_P^{(N)}/(4\pi)$.

VI. DISCUSSION AND CONCLUSION

The relation between the triple and quadruple couplings and the Regge residues is as follows:

(i) Triple-pomeron coupling: The relation between the pomeron coupling g_P and the residue of the pomeron is given by [7]

$$G_P^2 = \gamma_0^2(0)\left(\frac{\bar{s}}{\mathcal{M}^2}\right)^{\alpha_P(0)}\ , \tag{6.1}$$

where $\bar{s}\approx(6-8)\mathcal{M}^2$. Analogously, the relation between the triple-pomeron coupling g_{3P} and the triple-residue is given by

$$G_{3P} = r_0(0)\left(\frac{\bar{s}}{\mathcal{M}^2}\right)^{3\alpha_P(0)/2}\ . \tag{6.2}$$

Therefore,

$$\frac{G_{3P}}{G_P} = \frac{r_0(0)}{\gamma_0(0)}\left(\frac{\bar{s}}{\mathcal{M}^2}\right)^{\alpha_P(0)} \approx (6-8)\frac{r_0(0)}{\gamma_0(0)}\ . \tag{6.3}$$

According to [5] $r_0(0)/\gamma_0(0) = 1/40$ and therefore we expect $G_{3P}/G_P \approx (0.15 - 0.20)$. Comparing this with the result of the previous section implies that what is needed in the nuclear saturation is a factor two larger as expected from the triple-pomeron contribution. This leaves room for a contribution also from the change in the vector- (and scalar-) meson masses, which we used in [12].

(ii) Quadruple-pomeron coupling:
Similarly to the triple-pomeron vertex, taking the relation between the quadruple-pomeron coupling g_{4P} and the quadruple-residue q_0 as given by

$$G_{4P} = q_0(0)\left(\frac{\bar{s}}{\mathcal{M}^2}\right)^{2\alpha_P(0)} . \tag{6.4}$$

Then,

$$\frac{G_{4P}}{G_P} = \frac{q_0(0)}{\gamma_0(0)}\left(\frac{\bar{s}}{\mathcal{M}^2}\right)^{3\alpha_P(0)/2} \approx (14.5 - 22.5)\frac{q_0(0)}{\gamma_0(0)} . \tag{6.5}$$

(iii) Quadruple-pomeron in Reggeon field theory:
In Reggeon field theory, see e.g. [6], the (bare) gap Δ_0 of the pomeron intercept i.e. $\alpha_P(0) = 1 - \Delta_0$ and the (bare) triple- and quartic- couplings, respectively r_0 and λ_0, is related by $\Delta_0 = -r_0^2/\lambda_0$. For an estimate we identify: $g'_{3P} = r_0$ and $g'_{4P} = 4\lambda_0$. In comparing with Regge phenomenology of the total cross sections we do not distinguish here between 'bare' and 'renormalized' quantities. In fitting the high-energy pp cross sections, Donnachie and Landshoff [13] used the 'hard' and the 'soft' pomeron trajectories $\alpha_0(t)$ and $\alpha_1(t)$ respectively:

$$\begin{aligned} \alpha_0(t) &= 1 - \Delta_0 + \alpha' t, \\ \alpha_1(t) &= 1 - \Delta_1 + \alpha' t, \end{aligned}$$

For the soft pomeron they fitted $\Delta_1 = -0.0667$, and for the hard pomeron $\Delta_0 = -0.452$. Using the soft pomeron and the relation above from [6], we find

$$G_{4P} = -4r_0^2/\Delta_1 \approx 60 G_{3P}^2,$$

which gives $G_{4P}/4\pi \approx 30$ for $G_{3P}/4\pi = 0.2$. *So, apart from the precise numbers for the parameters the result seems to be that $G_{4P} >> G_{3P}$.*

Remark: *Also G_{3P} and G_{4P} are* **running** *coupling constants. Therefore for low energies these couplings may be larger than in the Regge-regime.*

(iv) Polynomial-pomeron coupling:
Consider a general polynomial pomeron-vertex, using the Lagrangian

$$\mathcal{L}_{Pol.} = \sum_{N=3}^{\infty} G_P^{(N)} \mathcal{M}^{4-N}\ \sigma_P^N(x). \tag{6.6}$$

Then, from the results above the effective two-body repulsion is given by

$$\begin{aligned} V_{eff}^{(Pol)}(\mathbf{x}_1, \mathbf{x}_2) &= \sum_{N=3}^{\infty}\left[g_P^{(N)} g_P^N \frac{\rho_{NM}^{N-2}}{\mathcal{M}^{3N-4}}\right] \cdot \frac{1}{4\pi}\frac{4}{\sqrt{\pi}}\left(\frac{m_P}{\sqrt{2}}\right)^3 \exp\left(-\frac{1}{2}m_P^2 r_{12}^2\right) \\ &\equiv \frac{1}{4\pi}\frac{4}{\sqrt{\pi}}\left(\frac{m_P}{\sqrt{2}}\right)^3 \exp\left(-\frac{1}{2}m_P^2 r_{12}^2\right) \cdot f(g_P, \rho_{MN}), \end{aligned} \tag{6.7}$$

with the volume-integral

$$I^{(N)}_{V,eff} = \sum_{N=3}^{\infty} g_P^{(N)} g_P^N \frac{\rho_{NM}^{N-2}}{\mathcal{M}^{3N-4}} = f(g_P, \rho_{NM}). \tag{6.8}$$

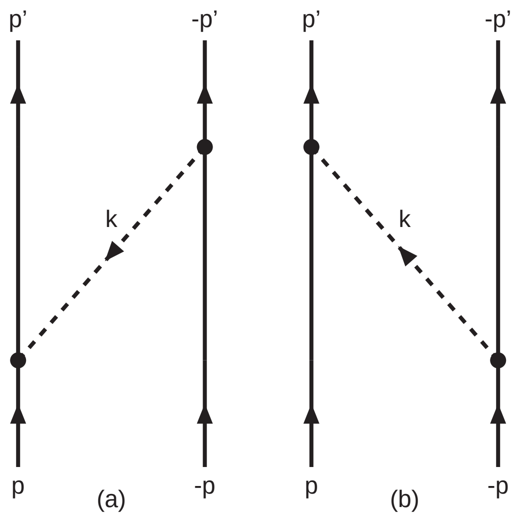

FIG. 2: CM One-boson-exchange graphs: The dashed lines with momentum **k** refers to the bosons: pseudo-scalar, vector, axial-vector, or scalar mesons.

APPENDIX A: DERIVATION CONFIGURATION-SPACE POTENTIALS

In Fig. 2 the two time-ordered graphs are drawn for a scalar exchange proces. In momentum space the matrix element from (a) and (b) is, realizing that two time-ordered graphs are equivalent to a single Feynman graph,

$$\langle p_1', p_2'|M|p_1, p_2\rangle \;=\; -G^2\; \delta^3(p_1' + p_2' - p_1 - p_2)\; \frac{1}{\omega_k^2}, \tag{A1}$$

where we used that in the CM-frame energy conservation makes the energy transfer zero, and the notation $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$.

Splitting off the CM-motion goes as follows. With

$$\mathbf{R} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) \quad , \; \mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2,$$
$$\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2) \quad , \; \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2,$$

the two-particle wave function is

$$(\mathbf{x}_1, \mathbf{x}_2|\mathbf{p}_1, \mathbf{p}_2) \;=\; \exp\left[i(\mathbf{p}_1 + \mathbf{p}_2)\cdot\mathbf{R}\right]\cdot\exp\left[\frac{i}{2}(\mathbf{p}_1 - \mathbf{p}_2)\cdot\mathbf{r}\right].$$

In configuration space

$$\langle x_1', x_2'|M|x_1, x_2\rangle = \int\frac{d^3p_1'd^3p_2'}{(2\pi)^6}\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\ (x_1'|p_1')(x_2'|p_2')(p_2|x_2)(p_1|x_1)\cdot$$
$$\times\langle p_1', p_2'|M|p_1, p_2\rangle = (2\pi)^{-12}\int d^3p_1'd^3p_2'\ \int d^3p_1d^3p_2\cdot$$
$$\times e^{-i(p_1'\cdot x_1'+p_2'\cdot x_2')}\ e^{+i(p_1\cdot x_1+p_2\cdot x_2)}\langle p_1', p_2'|M|p_1, p_2\rangle = (2\pi)^{-12}\cdot$$
$$\times\int d^3P'd^3p'\int d^3Pd^3p\ e^{-i(\mathbf{P}'\cdot\mathbf{R}'-\mathbf{P}\cdot\mathbf{R})}\ e^{-i(\mathbf{p}'\cdot\mathbf{r}'-\mathbf{p}\cdot\mathbf{r})}\ \langle\mathbf{p}', \mathbf{P}'|M|\mathbf{p}, \mathbf{P}\rangle. \quad \text{(A2)}$$

With

$$(\mathbf{p}', \mathbf{P}'|M|\mathbf{p}, \mathbf{P}) = \delta(\mathbf{P}'-\mathbf{P})\ M(\mathbf{p}', \mathbf{p})$$

Performing the P and P' integrations one obtains

$$\langle x_1', x_2'|M|x_1, x_2\rangle = (2\pi)^{-3}\delta(\mathbf{R}'-\mathbf{R})\ (\mathbf{r}'|M|\mathbf{r}), \quad \text{(A3a)}$$
$$(\mathbf{r}'|M|\mathbf{r}) = (2\pi)^{-6}\int\int d^3p'd^3p\ e^{-i(\mathbf{p}'\cdot\mathbf{r}'-\mathbf{p}\cdot\mathbf{r})}\ M(\mathbf{p}', \mathbf{p}). \quad \text{(A3b)}$$

Introducing the standard variables

$$\mathbf{q} = \frac{1}{2}(\mathbf{p}'+\mathbf{p})\ ,\ \mathbf{k} = \mathbf{p}'-\mathbf{p}, \quad \text{(A4)}$$

and replacing $\int d^3p'd^3p \rightarrow \int d^3qd^3k$, the q integrations can be executed immediately. One gets for $M(\mathbf{k}) = -G^2/\omega^2(\mathbf{k})$

$$(\mathbf{r}'|M|\mathbf{r}) = (2\pi)^{-6}\int\int d^3qd^3k\ e^{-i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})}e^{-i\mathbf{k}\cdot(\mathbf{r}'+\mathbf{r})/2}\ M(\mathbf{q}, \mathbf{k}) \quad \text{(A5a)}$$
$$\Rightarrow \int\frac{d^3k}{(2\pi)^3}\ e^{-i\mathbf{k}\cdot\mathbf{r}}\ M(\mathbf{k}). \quad \text{(A5b)}$$

For Pomeron exchange $-1/\omega^2 \rightarrow +\exp(-\mathbf{k}^2/\Lambda^2)/\mathcal{M}^2$. Then, one has with $\mathbf{r}_{12} = \mathbf{x}_1 - \mathbf{x}_2$,

$$\langle x_1', x_2'|M_P|x_1, x_2\rangle = (2\pi)^{-3}\delta(\mathbf{R}'-\mathbf{R})\ (\mathbf{r}'|V_P|\mathbf{r}_{12}),$$
$$V_P(r_{12}) = \frac{G^2}{4\pi}\frac{1}{2\pi\sqrt{\pi}}\ \frac{\Lambda^3}{\mathcal{M}^2}e^{-m_P^2r_{12}^2} = \frac{G^2}{4\pi}\frac{4}{\sqrt{\pi}}\ \frac{m_P^3}{\mathcal{M}^2}\ e^{-m_P^2r_{12}^2}. \quad \text{(A6)}$$

which explains Eq. 2.3.

APPENDIX B: THREE-BODY CONFIGURATION-SPACE POTENTIALS, JACOBIAN-COORDINATES METHOD

The free three-particle wave function is

$$\psi_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \Pi_{i=1}^3\left[e^{i\mathbf{p}_i\cdot\mathbf{x}_i}\right]. \quad \text{(B1)}$$

The matrix element corresponding to the triple-pomeron graph in Fig. 1 is

$$(p_1', p_2', p_3'|M|p_1, p_2, p_3) \;=\; G_{3P} G_P^3 \mathcal{M}\; \Pi_{i=1}^3 \left[\frac{e^{-\mathbf{k}_i^2/\Lambda^2}}{\mathcal{M}^2}\right] \left(\sum_i p_i' - \sum_i p_i\right), \tag{B2}$$

where $\mathbf{k}_i = \mathbf{p}_i' - \mathbf{p}_i$.

The Jacobi-coordinates in configuration and momentum space are defined as

$$\mathbf{x}_\rho = \frac{1}{\sqrt{2}}(\mathbf{x}_1 - \mathbf{x}_2)\ ,\ \mathbf{p}_\rho = \frac{1}{\sqrt{2}}(\mathbf{p}_1 - \mathbf{p}_2) \tag{B3a}$$

$$\mathbf{x}_\lambda = \frac{1}{\sqrt{6}}(\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3)\ ,\ \mathbf{p}_\lambda = \frac{1}{\sqrt{6}}(\mathbf{p}_1 + \mathbf{p}_2 - 2\mathbf{p}_3) \tag{B3b}$$

$$\mathbf{R}_3 = \frac{1}{\sqrt{3}}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)\ ,\ \mathbf{P}_3 = \frac{1}{\sqrt{3}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3). \tag{B3c}$$

One has

$$\sum_{i=1}^3 \mathbf{p}_i \cdot \mathbf{x}_i \;=\; \mathbf{p}_\rho \cdot \mathbf{x}_\rho + \mathbf{p}_\lambda \cdot \mathbf{x}_\lambda + \mathbf{P}_3 \cdot \mathbf{R}_3,$$

$$\sum_{i=1}^3 \mathbf{k}_i^2 \;=\; \mathbf{k}_\rho + \mathbf{k}_\lambda^2 + (\mathbf{P}_3' - \mathbf{P}_3)^2.$$

The potential is given by

$$\begin{aligned}
&(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3|V_3|\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \Pi_{i=1}^3 \left[\int d^3p_i' \int d^3p_i\right] \psi_3^*(\mathbf{x}_1', \mathbf{x}_2', \mathbf{x}_3')\ (p_1', p_2', p_3'|M_{3P}|p_1, p_2, p_3)\cdot \\
&\times \psi_3^*(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = (2\pi)^{-18} \int d^3P_3' d^3p_\rho' d^3p_\lambda' \int d^3P d^3p_\rho d^3p_\lambda \exp\left[-i(\mathbf{P}_3' \cdot \mathbf{R}_3' - \mathbf{P}_3 \cdot \mathbf{P}_3)\right]\cdot \\
&\times \exp\left[-i(\mathbf{p}_\rho' \cdot \mathbf{x}_\rho' - \mathbf{p}_\rho \cdot \mathbf{x}_\rho)\right] \exp\left[-i(\mathbf{p}_\lambda' \cdot \mathbf{x}_\lambda' - \mathbf{p}_\lambda \cdot \mathbf{x}_\lambda)\right]\cdot \\
&\times G_{3P} G_P^3 \left[\mathcal{M}^2\right]^{-3}\ \exp\left\{-(\mathbf{k}_\rho^2 + \mathbf{k}_\lambda^2)/\Lambda\right\}\cdot \\
&\times \exp\left\{-(\mathbf{P}_3' - \mathbf{P}_3)^2/\Lambda^2\right\}\ (3\sqrt{3})^{-1}\delta^3(\mathbf{P}_3' - \mathbf{P}_3).
\end{aligned} \tag{B4}$$

Since everything factorizes we can perform all integrals in an elementary way. The integrals are

$$\begin{aligned}
I_{CM} \;&=\; (2\pi)^{-3} \int d^3P_3' d^3P_3 \exp\left[-i(\mathbf{P}_3' \cdot \mathbf{R}_3' - \mathbf{P}_3 \cdot \mathbf{P}_3)\right] \exp\left\{-(\mathbf{P}_3' - \mathbf{P}_3)^2/\Lambda^2\right\}\cdot \\
&\quad \times \delta^3(\mathbf{P}_3' - \mathbf{P}_3) = \delta^3(\mathbf{R}_3' - \mathbf{R}_3)
\end{aligned} \tag{B5a}$$

$$\begin{aligned}
I_\rho \;&=\; (2\pi)^{-6} \int d^3p_\rho' d^3p_\rho \exp\left[-i(\mathbf{p}_\rho' \cdot \mathbf{x}_\rho' - \mathbf{p}_\rho \cdot \mathbf{x}_\rho)\right] \exp\left\{-\mathbf{k}_\rho^2)/\Lambda^2\right\} \\
&=\; \delta^3(\mathbf{x}_\rho' - \mathbf{x}_\rho)\ \left(\frac{\Lambda}{2\sqrt{\pi}}\right)^3 \exp\left[-\frac{1}{4}\Lambda^2 \mathbf{x}_\rho^2\right],
\end{aligned} \tag{B5b}$$

$$\begin{aligned}
I_\lambda \;&=\; (2\pi)^{-6} \int d^3p_\lambda' d^3p_\lambda \exp\left[-i(\mathbf{p}_\lambda' \cdot \mathbf{x}_\lambda' - \mathbf{p}_\lambda \cdot \mathbf{x}_\lambda)\right] \exp\left\{-\mathbf{k}_\lambda^2)/\Lambda^2\right\} \\
&=\; \delta^3(\mathbf{x}_\lambda' - \mathbf{x}_\lambda)\ \left(\frac{\Lambda}{2\sqrt{\pi}}\right)^3 \exp\left[-\frac{1}{4}\Lambda^2 \mathbf{x}_\lambda^2\right].
\end{aligned} \tag{B5c}$$

Separating the δ-functions by defining

$$(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3|V_3|\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \left[\Pi_{i=1}^3 \delta^3(\mathbf{x}'_i - \mathbf{x}_i)\right]\ V_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) \tag{B6}$$

the potential becomes

$$V_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)\ =\ (2\pi)^{-9} G_{3P} G_P^3 \mathcal{M} \left(\frac{\Lambda}{\mathcal{M}}\right)^6 \left(\frac{\pi}{\sqrt{3}}\right)^3 \exp\left[-\frac{1}{4}\Lambda^2(\mathbf{x}_\rho^2 + \mathbf{x}_\lambda^2)\right] \tag{B7}$$

Integration over particle 3 gives

$$V_{eff}(\mathbf{x}_1,\mathbf{x}_2) = \rho_{MN} \int d^3x_3\ V(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3). \tag{B8}$$

Translating the integrand back to the variables $\mathbf{x}_i, i = 1,2,3)$ we have

$$f_3\ \equiv\ \mathbf{x}_\rho^2 + \mathbf{x}_\lambda^2 = \frac{2}{3}\left(\mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2 - \mathbf{x}_1\cdot\mathbf{x}_2 - \mathbf{x}_1\cdot\mathbf{x}_3 - \mathbf{x}_2\cdot\mathbf{x}_3\right),$$

which leads to the $\mathbf{x}_3$-integral

$$\int d^3x_3 \exp\left[-\frac{1}{6}\Lambda^2\left\{\mathbf{x}_3^2 - \mathbf{x}_3\cdot(\mathbf{x}_1+\mathbf{x}_2)\right\}\right]\ =\ \left(\frac{6\pi}{\Lambda^2}\right)^{3/2} \exp\left[\frac{1}{24}\Lambda^2(\mathbf{x}_1+\mathbf{x}_2)^2\right]$$

giving

$$\begin{aligned} V_{eff}(\mathbf{x}_1,\mathbf{x}_2)\ &=\ (2\pi)^{-9/2} G_{3P} G_P^3 \rho_{MN} (2)^{-3} \frac{\Lambda^3}{\mathcal{M}^5} \exp\left[-\frac{1}{8}\Lambda^2(\mathbf{x}_1-\mathbf{x}_2)^2\right] \\ &=\ (2\pi)^{-9/2} G_{3P} G_P^3 \rho_{MN} \frac{m_P^3}{\mathcal{M}^5} \exp\left[-\frac{1}{2} m_P^2 r_{12}^2\right], \end{aligned} \tag{B9}$$

where we used $\Lambda = 2m_P$. Inserting the *rationalized couplings* g_P, g_{3P} defined by $G_P = \sqrt{4\pi} g_P$ and $G_{3P} = (4\pi)^{3/2} g_{3P}$ one has

$$V_{eff}(\mathbf{x}_1,\mathbf{x}_2)\ =\ g_{3P} g_P^3\ \frac{\rho_{MN}}{\mathcal{M}^5}\cdot\frac{2}{\pi}\ \frac{4}{\sqrt{\pi}}\cdot\left(\frac{m_P}{\sqrt{2}}\right)^3 \exp\left[-\frac{1}{2}m_P^2 r_{12}^2\right], \tag{B10}$$

This formula agrees with (3.7b)**!**

APPENDIX C: FOUR-BODY CONFIGURATION-SPACE POTENTIALS, JACOBIAN-COORDINATES METHOD

The free four-particle wave function is

$$\psi_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = \Pi_{i=1}^4\left[e^{i\mathbf{p}_i\cdot\mathbf{x}_i}\right]. \tag{C1}$$

The matrix element corresponding to the triple-pomeron graph in Fig. 1 is

$$(p'_1,p'_2,p'_3,p'_4|M|p_1,p_2,p_3,p_4)\ =\ G_{4P} G_P^4 \mathcal{M}\ \Pi_{i=1}^4\left[\frac{e^{-\mathbf{k}_i^2/\Lambda^2}}{\mathcal{M}^2}\right]\left(\sum_i p'_i - \sum_i p_i\right), \tag{C2}$$

where $\mathbf{k}_i = \mathbf{p}'_i - \mathbf{p}_i$.

The Jacobi-coordinates in configuration and momentum space are defined as

$$\mathbf{x}_\rho = \frac{1}{\sqrt{2}}(\mathbf{x}_1 - \mathbf{x}_2)\ ,\ \mathbf{p}_\rho = \frac{1}{\sqrt{2}}(\mathbf{p}_1 - \mathbf{p}_2) \tag{C3a}$$

$$\mathbf{x}_\lambda = \frac{1}{\sqrt{6}}(\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3)\ ,\ \mathbf{p}_\lambda = \frac{1}{\sqrt{6}}(\mathbf{p}_1 + \mathbf{p}_2 - 2\mathbf{p}_3) \tag{C3b}$$

$$\mathbf{x}_\mu = \frac{1}{\sqrt{12}}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 - 3\mathbf{x}_4)\ ,\ \mathbf{p}_\mu = \frac{1}{\sqrt{12}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 - 3\mathbf{p}_4) \tag{C3c}$$

$$\mathbf{R}_4 = \frac{1}{\sqrt{4}}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)\ ,\ \mathbf{P}_4 = \frac{1}{\sqrt{4}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4). \tag{C3d}$$

One has

$$\begin{aligned}
\sum_{i=1}^{4} \mathbf{p}_i \cdot \mathbf{x}_i &= \mathbf{p}_\rho \cdot \mathbf{x}_\rho + \mathbf{p}_\lambda \cdot \mathbf{x}_\lambda + \mathbf{p}_\mu \cdot \mathbf{x}_\mu + \mathbf{P}_4 \cdot \mathbf{R}_4, \\
\sum_{i=1}^{4} \mathbf{k}_i^2 &= \mathbf{k}_\rho + \mathbf{k}_\lambda^2 + +\mathbf{k}_\mu^2 + (\mathbf{P}'_4 - \mathbf{P}_4)^2.
\end{aligned}$$

The potential is given by

$$\begin{aligned}
&(\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}'_4 | V_4 | \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \Pi_{i=1}^4 \left[\int d^3p'_i \int d^3p_i\right] \psi_4^*(\mathbf{x}'_1, \mathbf{x}'_2, \mathbf{x}'_3, \mathbf{x}'_4) \cdot \\
&\times (p'_1, p'_2, p'_3, p'_4 | M_{4P} | p_1, p_2, p_3, p_4) \psi_4^*(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) = \\
&(2\pi)^{-24} \int d^3P'_3 d^3p'_\rho d^3p'_\lambda d^3p'_\mu \int d^3P d^3p_\rho d^3p_\lambda d^3p_\mu \exp\left[-i(\mathbf{P}'_4 \cdot \mathbf{R}'_4 - \mathbf{P}_4 \cdot \mathbf{P}_4)\right] \cdot \\
&\times \exp\left[-i(\mathbf{p}'_\rho \cdot \mathbf{x}'_\rho - \mathbf{p}_\rho \cdot \mathbf{x}_\rho)\right] \exp\left[-i(\mathbf{p}'_\lambda \cdot \mathbf{x}'_\lambda - \mathbf{p}_\lambda \cdot \mathbf{x}_\lambda)\right] \exp\left[-i(\mathbf{p}'_\mu \cdot \mathbf{x}'_\mu - \mathbf{p}_\mu \cdot \mathbf{x}_\mu)\right] \cdot \\
&\times G_{4P} G_P^4 \left[\mathcal{M}^2\right]^{-4}\ \exp\left\{-(\mathbf{k}_\rho^2 + \mathbf{k}_\lambda^2 + \mathbf{k}_\mu^2)/\Lambda\right\} \cdot \\
&\times \exp\left\{-(\mathbf{P}'_4 - \mathbf{P}_4)^2/\Lambda^2\right\}\ (4\sqrt{4})^{-1} \delta^3(\mathbf{P}'_4 - \mathbf{P}_4).
\end{aligned} \tag{C4}$$

Since everything factorizes we can perform all integrals in an elementary way. The integrals are

$$
\begin{aligned}
I_{CM} &= (2\pi)^{-3}\int d^3P_4'd^3P_4 \exp\left[-i(\mathbf{P}_4'\cdot\mathbf{R}_4' - \mathbf{P}_4\cdot\mathbf{P}_4)\right]\exp\left\{-(\mathbf{P}_4'-\mathbf{P}_4)^2/\Lambda^2\right\}\cdot \\
&\quad \times\delta^3(\mathbf{P}_4'-\mathbf{P}_4) = \delta^3(\mathbf{R}_4'-\mathbf{R}_4) \qquad \text{(C5a)} \\
I_\rho &= (2\pi)^{-6}\int d^3p_\rho' d^3p_\rho \exp\left[-i(\mathbf{p}_\rho'\cdot\mathbf{x}_\rho' - \mathbf{p}_\rho\cdot\mathbf{x}_\rho)\right]\exp\left\{-\mathbf{k}_\rho^2)/\Lambda^2\right\} \\
&= \delta^3(\mathbf{x}_\rho'-\mathbf{x}_\rho)\ \left(\frac{\Lambda}{2\sqrt{\pi}}\right)^3 \exp\left[-\frac{1}{4}\Lambda^2\mathbf{x}_\rho^2\right], \qquad \text{(C5b)} \\
I_\lambda &= (2\pi)^{-6}\int d^3p_\lambda' d^3p_\lambda \exp\left[-i(\mathbf{p}_\lambda'\cdot\mathbf{x}_\lambda' - \mathbf{p}_\lambda\cdot\mathbf{x}_\lambda)\right]\exp\left\{-\mathbf{k}_\lambda^2)/\Lambda^2\right\} \\
&= \delta^3(\mathbf{x}_\lambda'-\mathbf{x}_\lambda)\ \left(\frac{\Lambda}{2\sqrt{\pi}}\right)^3 \exp\left[-\frac{1}{4}\Lambda^2\mathbf{x}_\lambda^2\right], \qquad \text{(C5c)} \\
I_\mu &= (2\pi)^{-6}\int d^3p_\mu' d^3p_\mu \exp\left[-i(\mathbf{p}_\mu'\cdot\mathbf{x}_\mu' - \mathbf{p}_\mu\cdot\mathbf{x}_\mu)\right]\exp\left\{-\mathbf{k}_\mu^2)/\Lambda^2\right\} \\
&= \delta^3(\mathbf{x}_\mu'-\mathbf{x}_\mu)\ \left(\frac{\Lambda}{2\sqrt{\pi}}\right)^3 \exp\left[-\frac{1}{4}\Lambda^2\mathbf{x}_\mu^2\right]. \qquad \text{(C5d)}
\end{aligned}
$$

Separating the δ-functions by defining

$$
(\mathbf{x}_1',\mathbf{x}_2',\mathbf{x}_3',\mathbf{x}_4'|V_4|\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \left[\Pi_{i=1}^4\delta^3(\mathbf{x}_i'-\mathbf{x}_i)\right]\ V_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \qquad \text{(C6)}
$$

the potential becomes

$$
\begin{aligned}
V_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) &= (2\pi)^{-12}G_{4P}G_P^4\ \mathcal{M}\ \left(\frac{\Lambda}{\mathcal{M}}\right)^9\ (\pi)^{9/2}\cdot \\
&\quad \times\exp\left[-\frac{1}{4}\Lambda^2(\mathbf{x}_\rho^2+\mathbf{x}_\lambda^2+\mathbf{x}_\mu^2)\right]. \qquad \text{(C7)}
\end{aligned}
$$

Integration over particle 3 and 4 gives

$$
V_{eff}(\mathbf{x}_1,\mathbf{x}_2) = \rho_{MN}^2\int d^3x_3d^3x_4\ V(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4). \qquad \text{(C8)}
$$

Translating the integrand back to the variables $\mathbf{x}_i, i=1,2,3)$ we have

$$
\begin{aligned}
f_4 \equiv\ & \mathbf{x}_\rho^2+\mathbf{x}_\lambda^2+\mathbf{x}_\mu^2 = \frac{3}{4}\left(\mathbf{x}_1^2+\mathbf{x}_2^2+\mathbf{x}_3^2+\mathbf{x}_4^2\right) \\
& -\frac{1}{2}\left(\mathbf{x}_1\cdot\mathbf{x}_2+\mathbf{x}_1\cdot\mathbf{x}_3+\mathbf{x}_2\cdot\mathbf{x}_3\mathbf{x}_1\cdot\mathbf{x}_4+\mathbf{x}_2\cdot\mathbf{x}_4+\mathbf{x}_3\cdot\mathbf{x}_4\right) \\
=\ & \frac{3}{4}(\mathbf{x}_1^2+\mathbf{x}_2^2-\frac{2}{3}\mathbf{x}_1\cdot\mathbf{x}_2)+\frac{3}{4}(\mathbf{x}_3^2+\mathbf{x}_4^2) \\
& -\frac{1}{2}\left[(\mathbf{x}_1+\mathbf{x}_2)\cdot(\mathbf{x}_3+\mathbf{x}_4)+\mathbf{x}_3\cdot\mathbf{x}_4\right] \\
=\ & \frac{3}{4}(\mathbf{x}_1^2+\mathbf{x}_2^2-\frac{2}{3}\mathbf{x}_1\cdot\mathbf{x}_2)+\frac{1}{2}(\mathbf{x}_3-\mathbf{x}_4)^2 \\
& +\frac{1}{4}(\mathbf{x}_3+\mathbf{x}_4)^2-\frac{1}{2}(\mathbf{x}_1+\mathbf{x}_2)\cdot(\mathbf{x}_3+\mathbf{x}_4).
\end{aligned}
$$

Introducing $\mathbf{x} = (\mathbf{x}_3 + \mathbf{x}_4)/2$ and $\mathbf{y} = \mathbf{x}_3 - \mathbf{x}_4$ leads to the 34-integrals

$$\int d^3x d^3y \exp\left[-\frac{1}{4}\Lambda^2\left\{\mathbf{x}^2 - \mathbf{x}\cdot(\mathbf{x}_1+\mathbf{x}_2)\right\} - \frac{1}{8}\Lambda^2\mathbf{y}^2\right] =$$
$$\left(\frac{8\pi}{\Lambda^2}\right)^{3/2}\left(\frac{4\pi}{\Lambda^2}\right)^{3/2}\exp\left[\frac{1}{16}\Lambda^2(\mathbf{x}_1+\mathbf{x}_2)^2\right]$$

giving

$$\begin{aligned} V_{eff}(\mathbf{x}_1,\mathbf{x}_2) &= (2\pi)^{-9/2}G_{4P}G_P^4\rho_{MN}^2\frac{\Lambda^3}{\mathcal{M}^8}\exp\left[-\frac{1}{8}\Lambda^2(\mathbf{x}_1-\mathbf{x}_2)^2\right] \\ &= (2\pi)^{-9/2}G_{4P}G_P^4\rho_{MN}^2\ (2\sqrt{2})\ \frac{m_P^3}{\mathcal{M}^8}\exp\left[-\frac{1}{2}m_P^2r_{12}^2\right], \end{aligned} \tag{C9}$$

where we used $\Lambda = 2m_P$. Inserting the *rationalized couplings* g_P, g_{4P} defined by $G_P = \sqrt{4\pi}g_P$ and $G_{4P} = (4\pi)^2 g_{4P}$ one has

$$V_{eff}(\mathbf{x}_1,\mathbf{x}_2) = 8g_{4P}g_P^4\ \frac{\rho_{MN}^2}{\mathcal{M}^8}\cdot\frac{4}{\sqrt{\pi}}\ \left(\frac{m_P}{\sqrt{2}}\right)^3\exp\left[-\frac{1}{2}m_P^2r_{12}^2\right], \tag{C10}$$

This formula agrees with (4.7a)**!**

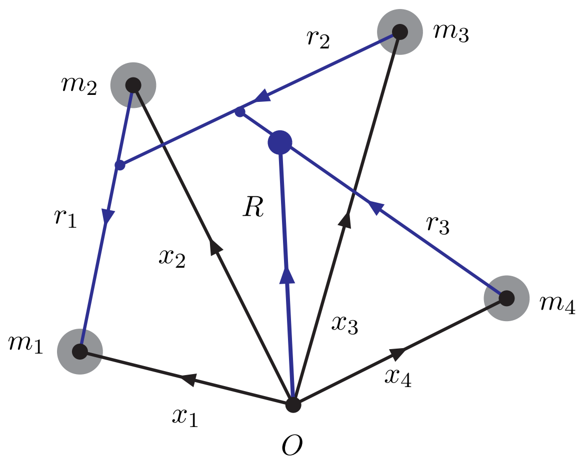

FIG. 3: *Jacobi-coordinates of a four particle system.*

APPENDIX D: JACOBI-COORDINATES A=4 SYSTEMS

For an N-body system the Jacobian coordinates $\mathbf{r}_i$ are constructed via the following rules:

$$\mathbf{r}_1 = \mathbf{x}_1 - \mathbf{x}_2, \tag{D1a}$$
$$\mathbf{r}_j = \sum_{k=1}^{j}\frac{m_k}{m_{0j}}\mathbf{x}_k - \mathbf{x}_{j+1},\quad m_{0j} = \sum_{k=1}^{j} m_k. \tag{D1b}$$

Here, $\mathbf{x}_{N+1} = 0$ and for j=N this is defined as $\mathbf{r}_N \equiv \mathbf{R}$ the center of mass

$$\mathbf{R} = \frac{1}{M}\sum_{k=1} m_k \mathbf{x}_k, \; M = m_{0N} = \sum_{k=1} m_k. \tag{D2}$$

For N=4 this leads to the Jacobian coordinates

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{x}_1 - \mathbf{x}_2, && \text{(D3a)}\\
\mathbf{r}_2 &= \mathbf{R}_{12} - \mathbf{x}_3 = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1+m_2} - \mathbf{x}_3, && \text{(D3b)}\\
\mathbf{r}_3 &= \mathbf{R}_{123} - \mathbf{x}_4 = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + m_3\mathbf{x}_3}{m_1+m_2+m_3} - \mathbf{x}_4, && \text{(D3c)}\\
\mathbf{R} &= \mathbf{R}_{1234} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2 + m_3\mathbf{x}_3 + m_4\mathbf{x}_4}{m_1+m_2+m_3+m_4}. && \text{(D3d)}
\end{aligned}$$

The inverse of (D3) reads

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{R} + \frac{m_2}{m_1+m_2}\mathbf{r}_1 + \frac{m_3}{m_1+m_2+m_3}\mathbf{r}_2 + \frac{m_4}{m_1+m_2+m_3+m_4}\mathbf{r}_3, && \text{(D4a)}\\
\mathbf{x}_2 &= \mathbf{R} - \frac{m_1}{m_1+m_2}\mathbf{r}_1 + \frac{m_3}{m_1+m_2+m_3}\mathbf{r}_2 + \frac{m_4}{m_1+m_2+m_3+m_4}\mathbf{r}_3, && \text{(D4b)}\\
\mathbf{x}_3 &= \mathbf{R} - \frac{m_1+m_2}{m_1+m_2+m_3}\mathbf{r}_2 + \frac{m_4}{m_1+m_2+m_3+m_4}\mathbf{r}_3, && \text{(D4c)}\\
\mathbf{x}_4 &= \mathbf{R} - \frac{m_1+m_2+m_3}{m_1+m_2+m_3+m_4}\mathbf{r}_3. && \text{(D4d)}
\end{aligned}$$

1. Four-pomeron Potential

For the multi-pomeron potentials for the leading term we neglect the baryon mass-differences. Therefore we take $m_1 = m_2 = m_3 = m_4$. Then,

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{x}_1 - \mathbf{x}_2 = \sqrt{2}\mathbf{x}_\rho, && \text{(D5a)}\\
\mathbf{r}_2 &= \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) - \mathbf{x}_3 = \sqrt{\frac{3}{2}}\mathbf{x}_\lambda, && \text{(D5b)}\\
\mathbf{r}_3 &= \frac{1}{3}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) - \mathbf{x}_4 = \sqrt{\frac{4}{3}}\mathbf{x}_\mu, && \text{(D5c)}\\
\mathbf{r}_4 &= \frac{1}{4}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4) = \sqrt{\frac{1}{4}}\mathbf{R}, && \text{(D5d)}
\end{aligned}$$

with the inverse

$$\begin{aligned}
\mathbf{x}_1 &= \mathbf{R} + \frac{1}{2}\mathbf{r}_1 + \frac{1}{3}\mathbf{r}_2 + \frac{1}{4}\mathbf{r}_3, && \text{(D6a)}\\
\mathbf{x}_2 &= \mathbf{R} - \frac{1}{2}\mathbf{r}_1 + \frac{1}{3}\mathbf{r}_2 + \frac{1}{4}\mathbf{r}_3, && \text{(D6b)}\\
\mathbf{x}_3 &= \mathbf{R} - \frac{2}{3}\mathbf{r}_2 + \frac{1}{4}\mathbf{r}_3, && \text{(D6c)}\\
\mathbf{x}_4 &= \mathbf{R} - \frac{3}{4}\mathbf{r}_3. && \text{(D6d)}
\end{aligned}$$

Analogous to the A=3 case we work with the configuration and momentum space Jacobi-variables

$$\mathbf{x}_\rho = \frac{1}{\sqrt{2}}(\mathbf{x}_1 - \mathbf{x}_2)\ \ ,\ \ \mathbf{p}_\rho = \frac{1}{\sqrt{2}}(\mathbf{p}_1 - \mathbf{p}_2), \tag{D7a}$$

$$\mathbf{x}_\lambda = \frac{1}{\sqrt{6}}(\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3)\ \ ,\ \ \mathbf{p}_\lambda = \frac{1}{\sqrt{6}}(\mathbf{p}_1 + \mathbf{p}_2 - 2\mathbf{p}_3), \tag{D7b}$$

$$\mathbf{x}_\mu = \frac{1}{\sqrt{12}}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 - 3\mathbf{x}_4)\ \ ,\ \ \mathbf{p}_\mu = \frac{1}{\sqrt{12}}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 - 3\mathbf{p}_4), \tag{D7c}$$

$$\mathbf{R} = \frac{1}{4}(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)\ \ ,\ \ \mathbf{P} = \frac{1}{4}(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4). \tag{D7d}$$

This gives

$$\sum_{i=1}^{4} \mathbf{p}_i = \mathbf{P}\cdot\mathbf{R} + \mathbf{p}_\rho\cdot\mathbf{x}_\rho + \mathbf{p}_\lambda\cdot\mathbf{x}_\lambda + \mathbf{p}_\mu\cdot\mathbf{x}_\mu \tag{D8}$$

The connection with the Jacobi-coordinates used in the case of the triton is given by

$$\mathbf{r}_1 = \boldsymbol{\rho},\ \ \ \mathbf{r}_2 = \sqrt{\frac{3}{2}}\ \boldsymbol{\lambda}, \tag{D9}$$

which indeed yields

$$(\mathbf{x}_1 - \mathbf{x}_2)^2 + (\mathbf{x}_1 - \mathbf{x}_3)^2 + (\mathbf{x}_2 - \mathbf{x}_3)^2 = 3(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2).$$

In Fig. 3 the constellation of the different vectors are displayed. We note that only particle 4 is connected with the center of mass.

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$$\int d^3x\ |\mathbf{x}\rangle\langle\mathbf{x}| = 1\ \ \ ,\ \int \frac{d^3p}{(2\pi)^3}\ |\mathbf{p}\rangle\langle\mathbf{p}| = 1$$

and the relation of matrix elements in configuration and momentum space reads

$$
\begin{aligned}
(\mathbf{r}'|V|\mathbf{r}) &= \iint \frac{d^3p'd^3p}{(2\pi)^6}\ (\mathbf{p}'|V|\mathbf{p})(\mathbf{p}|\mathbf{r}) \\
&= \iint \frac{d^3qd^3k}{(2\pi)^6}\ e^{i(\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r})}e^{i(\mathbf{k}\cdot(\mathbf{r}'+\mathbf{r})/2}V(\mathbf{q},\mathbf{k})
\end{aligned}
$$

where $\mathbf{q} = (\mathbf{p}' + \mathbf{p})/2, \mathbf{k} = \mathbf{p}' - \mathbf{p}$.

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